
Strongly Hyperbolic Extensions of the ADM Hamiltonian

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Summary. The ADM Hamiltonian formulation of general relativity with prescribed lapse and shift is a weakly hyperbolic system of partial differential equations. In general weakly hyperbolic systems are not mathematically well posed. For well posedness, the theory should be reformulated so that the complete system, evolution equations plus gauge conditions, is (at least) strongly hyperbolic. Traditionally, reformulation has been carried out at the level of equations of motion. This typically destroys the variational and Hamiltonian structures of the theory. Here I show that one can extend the ADM formalism to (i) incorporate the gauge conditions as dynamical equations and (ii) affect the hyperbolicity of the complete system, all while maintaining a Hamiltonian description. The extended ADM formulation is used to obtain a strongly hyperbolic Hamiltonian description of Einstein's theory that is generally covariant under spatial diffeomorphisms and time reparametrizations, and has physical characteristics. The extended Hamiltonian formulation with 1+log slicing and gamma-driver shift conditions is weakly hyperbolic.

1 Introduction

This paper is dedicated to Claudio Bunster in celebration of his sixtieth birthday. In a remarkable body of work Claudio showed that we can view the Hamiltonian formulation of general relativity as fundamental. (See in particular references [26, 14, 27].) He considered the requirement that the sequence of spatial three-geometries evolved by the Hamiltonian should be interpretable as a four-dimensional spacetime. From this assertion and a few modest assumptions he was able to derive the ADM Hamiltonian [8, 1] of general relativity. A number of deep insights into the nature of gravity and matter came from his analysis, including the role of gauge symmetries in electrodynamics and Yang–Mills theories and the necessity for all matter fields to couple to gravity.

In general relativity we are faced with the practical problem of predicting the future evolution of strongly gravitating systems, including interacting black holes and neutron stars. Such problems fall into the realm of numerical relativity. Naturally, the first attempts at numerical modeling in general relativity were based on the ADM Hamiltonian equations. By the early 1990's it became clear that the ADM equations were not appropriate for numerical computation because they are not well posed in a mathematical sense. What followed was more than a decade of activity in which the ADM equations were rewritten in a variety of ways. The goal

was to produce a well posed system of partial differential equations (PDE's) for Einstein's theory. One strategy for modifying the ADM equations was to add multiples of the constraints to the right-hand sides. Another strategy was to introduce new independent variables defined as combinations of metric tensor components and their spatial derivatives. This later strategy introduces new constraints into the system, namely, the constraints that the definitions of the new variables should hold for all time.

In a practical sense, the effort to re-express the ADM equations has been successful. Currently there are a number of formulations of the Einstein evolution equations that appear to be "good enough", the most popular for numerical work being the BSSN system.[24, 2] BSSN relies on a conformal splitting of the metric and extrinsic curvature. It introduces new independent variables, the "conformal connection functions", defined as the trace (in its lower indices) of the Christoffel symbols built from the conformal metric.

BSSN and the other "modern" formulations of Einstein's theory are very clever. But at a basic level, these formulations are obtained through a manipulation of the equations of motion as a system of PDE's. What is invariably lost is the beautiful Hamiltonian structure found in the ADM formulation. In this paper I present a systematic procedure that can be used to modify the ADM equations in an effort to obtain a good system of PDE's without losing the Hamiltonian structure.

A good system of PDE's is one that is mathematically well posed. As a general rule, a system formulated in space without boundaries must be strongly hyperbolic to be well posed. If boundaries are present, an even stronger notion of hyperbolicity, symmetric hyperbolicity, is needed to prove well posedness. We are interested in extensions of the ADM equations that, like the ADM equations themselves, have first-order time and second-order space derivatives. It turns out that a simple prescription can be given to test for strong hyperbolicity in such systems of PDE's. The justification for this prescription requires a rather deep mathematical analysis, but the prescription itself is fairly easy to apply. In section II, I discuss hyperbolicity and justify the prescription for well posedness with heuristic arguments.

Another issue that has become apparent from recent numerical work is the benefit, in practice, of incorporating the slicing and coordinate conditions (gauge conditions) as dynamical equations. That is, the lapse function and shift vector are not fixed a priori but are determined along with the other fields through evolution equations of their own. The hyperbolicity of the entire system of PDE's, including the equations for the lapse and shift, must be considered. The issues of gauge conditions and hyperbolicity cannot be separated.

In this paper I show that the Hamiltonian formulation of general relativity can be extended to (i) incorporate dynamical gauge conditions and (ii) alter the level of hyperbolicity. In section III the ADM formulation is enlarged by the introduction of momentum variables π and p_a conjugate to the lapse function α and shift vector β^a . In this way the lapse and shift become dynamical. The new momenta are primary constraints and they appear in the action with undetermined multipliers Λ and Ω^a . The usual Hamiltonian and momentum constraints, \mathcal{H} and \mathcal{M}_a , are secondary constraints. This Hamiltonian formulation of Einstein's theory is not new [7, 12], and is not substantially different from the original ADM formulation—like the ADM formulation, it is only weakly hyperbolic. This is shown in section IV.

The Hamiltonian formulation with dynamical lapse and shift is extended in section V by allowing the multipliers Λ and Ω^a to depend on the canonical variables. This has two effects. First, it changes the evolution equations for the lapse and shift, yielding gauge conditions that depend on the dynamical variables. Second, it changes the principal parts of the evolution equations and potentially changes the level of hyperbolicity of the system. The hyperbolicity of the extended Hamiltonian formulation is considered in section VI for a fairly general choice of multipliers that preserves spatial diffeomorphism invariance and time reparametriza-

tion invariance. When the multipliers are chosen so that the evolution equations are strongly hyperbolic with physical characteristics, the system is equivalent in its principal parts to the generalized harmonic formulation of gravity.[9, 20, 16] It is also shown that the extended Hamiltonian formulation with 1+log slicing and the gamma-driver shift condition is only weakly hyperbolic. A few final remarks are presented in section VII.

2 Strong Hyperbolicity for Quasilinear Hamiltonian systems

Let q_μ, p_μ denote pairs of canonically conjugate fields. We will consider Hamiltonian systems for which Hamilton's equations are a quasilinear system of partial differential equations (PDE's). Thus we assume that the Hamiltonian H is a linear combination of terms that are at most quadratic in the momenta and spatial derivatives of the coordinates. More precisely, H should be a linear combination of terms $p_\mu p_\nu, (\partial_a q_\mu)(\partial_b q_\nu), p_\mu(\partial_a q_\nu), p_\mu, (\partial_a q_\mu)$, and 1 with coefficients that depend on the q 's.¹ (Here, ∂_a denotes the derivative with respect to the spatial coordinates.) One would like to show that Hamilton's equations are well posed as a system of PDE's. The subject of well posedness is a large, active area of research in mathematics and physics. In this section I present a very pedestrian account of the subject in the context of Hamiltonian field theory. Much more rigorous and complete discussions can be found elsewhere. (See, for example, references [11, 22, 25, 6, 23, 19, 10, 15].)

A well posed system is one whose solutions depend continuously on the initial data. For a well posed system, two sets of initial data that are close to one another will evolve into solutions that remain close for some finite time. A system is not well posed if it supports modes whose growth rates increase without bound with increasing wave number. A concrete example is given below.

In analyzing well posedness we are primarily concerned with the evolution in time of high wave number (short wavelength) perturbations of the initial data. For this purpose we can approximate the quasilinear system of PDE's by linearizing about a solution. That is, we expand the Hamiltonian to quadratic order in perturbations, which we denote $\delta q_\mu, \delta p_\mu$. We then look for Fourier modes of the form $\delta q_\mu = \bar{q}_\mu e^{i\omega t + ik_a x^a} / (i|k|)$, $\delta p_\mu = \bar{p}_\mu e^{i\omega t + ik_a x^a}$ with nonzero wave number k_a . Here, $|k| \equiv \sqrt{h^{ab} k_a k_b}$ is the norm of k_a defined in terms of a convenient metric h^{ab} (which could be the inverse of the physical spatial metric). If the ansatz for $\delta q_\mu, \delta p_\mu$ is substituted into the linearized Hamilton's equations, the system becomes

$$\omega v = (|k|A - iB - C/|k|)v \quad (1)$$

where v is the column vector $v = (\bar{q}_1, \bar{q}_2, \dots, \bar{p}_1, \bar{p}_2, \dots)^T$ of Fourier coefficients. Equation (1) shows that the problem of finding perturbative modes with wave number k_a is equivalent to the eigenvalue problem for the matrix $(|k|A - iB - C/|k|)$. The eigenvector is v and the eigenvalue is the frequency ω .

What one is really doing in the construction above is replacing the system of PDE's with a pseudo-differential system. The factor of $i|k|$ in the denominator of δq_μ is, in effect, equivalent to a change of variables in which q_μ is replaced by $q_\mu / \sqrt{h^{ab} \partial_a \partial_b}$. In this way the second order (in space derivatives) PDE's are replaced with an equivalent first order pseudo-differential system.[19]

¹ Note that terms proportional to $\partial_a p_\mu$ are also allowed in H since they are related to terms of the form $p_\mu(\partial_a q_\nu)$ through integration by parts.

The behavior of ω as $|k|$ becomes large depends on the leading order term A in the matrix $(|k|A - iB - C/|k|)$. The term A is the “principal symbol” of the system. It is constructed from the coefficients of the highest “weight” terms in the Hamiltonian, namely, the terms proportional to $p_\mu p_\nu$, $(\partial_a q_\mu)(\partial_b q_\nu)$, $p_\mu(\partial_a q_\nu)$ and $(\partial_a p_\mu)$. Note that it is not necessary to linearize the equations of motion (or expand the Hamiltonian to quadratic order) in order to find A . In practice we don’t actually linearize, we simply identify the coefficients of the highest weight terms in the PDE’s to form the matrix A .

If A has real eigenvalues and a complete set of eigenvectors that have smooth dependence on the unit vector $n_a \equiv k_a/|k|$, the system is said to be *strongly hyperbolic*. If A has real eigenvalues but the eigenvectors are not complete, the system is said to be *weakly hyperbolic*. It can be proved that a strongly hyperbolic system of quasilinear PDE’s is well-posed.[19]

Here is the rough idea. Let S denote the matrix whose rows are the left eigenvectors of A . Assuming strong hyperbolicity, the eigenvectors are complete and S^{-1} exists. The eigenvalue problem (1) can be written as $\omega \hat{v} = (|k|\hat{A} - i\hat{B} - \hat{C}/|k|)\hat{v}$ where $\hat{v} \equiv S v$, $\hat{A} \equiv SAS^{-1}$, $\hat{B} \equiv SBS^{-1}$, and $\hat{C} \equiv SCS^{-1}$. Note that \hat{A} is diagonal with entries equal to the (real) eigenvalues. Let a dagger (\dagger) denote the Hermitian conjugate (complex conjugate $*$ plus transpose T). Since $\hat{A}^\dagger = \hat{A}$, we find

$$\begin{aligned} (\omega - \omega^*)\hat{v}^\dagger \hat{v} &= \hat{v}^\dagger (\omega \hat{v}) - (\omega \hat{v})^\dagger \hat{v} \\ &= \hat{v}^\dagger (|k|\hat{A} - i\hat{B} - \hat{C}/|k|)\hat{v} - \hat{v}^\dagger (|k|\hat{A} + i\hat{B}^T - \hat{C}^T/|k|)\hat{v} \\ &= -i\hat{v}^\dagger (M + M^\dagger)\hat{v} \end{aligned} \quad (2)$$

where $M \equiv \hat{B} - i\hat{C}/|k|$. The left-hand side includes the factor $(\omega - \omega^*) = 2i\Im \omega = -2i\Re(i\omega)$, so Eq. (2) can be written as $2\Re(i\omega) = \hat{v}^\dagger (M + M^\dagger)\hat{v}/\hat{v}^\dagger \hat{v}$. It follows that $\Re(i\omega) \leq \tau^{-1}$ where $2\tau^{-1}$ is the maximum over $|k|$ of the matrix norm of $(M + M^\dagger)$. [The matrix norm is the maximum of the real number $\hat{v}^\dagger (M + M^\dagger)\hat{v}/(\hat{v}^\dagger \hat{v})$.] From this argument we see that the growth rate for the mode k_a is bounded; it can grow no faster than $e^{t/\tau}$ where τ is independent of k_a .

Consider a simple example in one spatial dimension with two pairs of canonically conjugate fields, q_1 , p_1 and q_2 , p_2 . Let the Hamiltonian be given by

$$\begin{aligned} H = \int dx \left\{ \frac{1}{2}[(p_1)^2 + (p_2)^2 + (q_1')^2 + (q_2')^2] + 2p_2 q_1' + 2p_1 q_2' \right. \\ \left. + p_1(q_2 + q_1) + p_2(q_2 - q_1) + q_1 q_2 \right\}. \end{aligned} \quad (3)$$

Hamilton’s equations are

$$\dot{q}_1 = p_1 + 2q_2' + q_2 + q_1, \quad (4a)$$

$$\dot{q}_2 = p_2 + 2q_1' + q_2 - q_1, \quad (4b)$$

$$\dot{p}_1 = q_1'' + 2p_2' + p_2 - p_1 - q_2, \quad (4c)$$

$$\dot{p}_2 = q_2'' + 2p_1' - p_1 - p_2 - q_1. \quad (4d)$$

In this example the PDE’s are linear so the linearization step is trivial: $q_\mu \rightarrow \delta q_\mu$, $p_\mu \rightarrow \delta p_\mu$. Now insert the ansatz $\delta q_\mu = \bar{q}_\mu e^{i\omega t + ikx}/(i|k|)$, $\delta p_\mu = \bar{p}_\mu e^{i\omega t + ikx}$. This yields

$$\omega \bar{q}_1 = |k|(2n\bar{q}_2 + \bar{p}_1) - i(\bar{q}_1 + \bar{q}_2), \quad (5a)$$

$$\omega \bar{q}_2 = |k|(2n\bar{q}_1 + \bar{p}_2) - i(\bar{q}_2 - \bar{q}_1), \quad (5b)$$

$$\omega \bar{p}_1 = |k|(\bar{q}_1 + 2n\bar{p}_2) - i(\bar{p}_2 - \bar{p}_1) + \bar{q}_2/|k|, \quad (5c)$$

$$\omega \bar{p}_2 = |k|(\bar{q}_2 + 2n\bar{p}_1) + i(\bar{p}_1 + \bar{p}_2) + \bar{q}_1/|k|, \quad (5d)$$

where $n \equiv k/|k|$ is the sign of the wave number k . Collecting the unknowns into a column vector $v = (\bar{q}_1, \bar{q}_2, \bar{p}_1, \bar{p}_2)^T$, we see that these equations become $\omega v = (|k|A - iB - C/|k|)v$ where the matrices are given by

$$A = \begin{pmatrix} 0 & 2n & 1 & 0 \\ 2n & 0 & 0 & 1 \\ 1 & 0 & 0 & 2n \\ 0 & 1 & 2n & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \quad (6)$$

The principal symbol A has real eigenvalues ± 1 , ± 3 , and a complete set of eigenvectors. Therefore this system is strongly hyperbolic. The modes with wave number k have frequencies $\omega = \pm|k| + \mathcal{O}(1/|k|)$ and $\omega = \pm 3|k| + \mathcal{O}(1/|k|)$. In particular the imaginary parts of the ω 's do not grow with increasing $|k|$.

Now replace the terms $2p_2q_1' + 2p_1q_2'$ in the Hamiltonian with $p_2q_1' - p_1q_2'$. The principal symbol becomes

$$A = \begin{pmatrix} 0 & -n & 1 & 0 \\ n & 0 & 0 & 1 \\ 1 & 0 & 0 & n \\ 0 & 1 & -n & 0 \end{pmatrix} \quad (7)$$

while B and C are unchanged. The eigenvalues of A vanish and there are only two independent eigenvectors. Therefore this system is weakly hyperbolic. The modes with wave number k have frequencies $\omega = \pm i\sqrt{2|k|} + \mathcal{O}(1)$ and $\omega = \pm\sqrt{2|k|} + \mathcal{O}(1)$. The modes with frequency $\omega \approx -i\sqrt{2|k|}$ will grow in time at a rate that is unbounded as $|k|$ increases.

The system described by this last example is not well posed. Indeed, consider two initial data sets that differ from one another by terms $q_\mu \sim \bar{q}_\mu e^{ikx}/|k|^2$, $p_\mu \sim \bar{p}_\mu e^{ikx}/|k|$ where $(\bar{q}_1, \bar{q}_2, \bar{p}_1, \bar{p}_2)^T$ is an eigenvector with eigenvalue $\omega = -i\sqrt{2|k|} + \mathcal{O}(1)$. In the limit as $|k| \rightarrow \infty$ these terms vanish and the two initial data sets coincide. However, if we evolve these data sets the solutions will differ at finite time t by terms $q_\mu \sim \bar{q}_\mu e^{\sqrt{2|k|}t + ikx}/|k|^2$, $p_\mu \sim \bar{p}_\mu e^{\sqrt{2|k|}t + ikx}/|k|$. These terms do not vanish in the limit $|k| \rightarrow \infty$. This system is ill posed because the solution at finite time does not depend continuously on the initial data.

In some cases it may be possible to model a physical system with ill posed PDE's and to gain important physical insights through a formal analysis. Claudio's beautiful work on the (weakly hyperbolic) ADM equations is a perfect example! One can imagine that the initial data are analytic, in which case the Cauchy–Kowalewski theorem guarantees that a solution exists for a finite time. But most data, even smooth data, are not analytic. From a computational point of view, having an ill posed system is unacceptable. Numerical errors will always introduce modes with large wave numbers, with the size of $|k|$ limited only by the details of the numerical implementation. For example, with a finite difference algorithm the maximum $|k|$ is roughly $1/\Delta x$ where Δx is the grid spacing. In practice it does not take long for the numerical solution to become dominated by this highest-wave number mode. As the grid resolution is increased (Δx is decreased), the unwanted highest wave number mode grows even more quickly. For practical numerical studies, we need our system of PDE's to be well posed.

The analysis outlined above leads to the following test for strong hyperbolicity. We begin by constructing the principal symbol A from the principal parts of Hamilton's equations. The principal parts of the \dot{q}_μ equations are the terms proportional to p_μ and $\partial_a q_\mu$. In these terms we make the replacements $p_\mu \rightarrow \bar{p}_\mu$ and $\partial_a q_\mu \rightarrow n_a \bar{q}_\mu$. The principal parts of the \dot{p}_μ equations are the terms proportional to $\partial_a p_\mu$ and $\partial_a \partial_b q_\mu$. In these terms we make the replacements $\partial_a p_\mu \rightarrow n_a \bar{p}_\mu$ and $\partial_a \partial_b q_\mu \rightarrow n_a n_b \bar{q}_\mu$. The principal symbol A is the matrix formed from the coefficients of the \bar{q} 's and \bar{p} 's. The next step is to compute the eigenvalues and eigenvectors

of A . If A has real eigenvalues and a complete set of eigenvectors that depend smoothly on n_a , the system is strongly hyperbolic and the initial value problem is well posed.

3 ADM with dynamical lapse and shift

The Einstein–Hilbert action is $S = \int d^4x \sqrt{-\mathbf{g}} \mathbf{R}$ where \mathbf{g} is the determinant of the spacetime metric and \mathbf{R} is the spacetime curvature scalar. Units are chosen such that Newton’s constant equals $1/(16\pi)$. With the familiar splitting of the spacetime metric into the spatial metric g_{ab} , lapse function α , and shift vector β^a , the action becomes

$$S[g, \alpha, \beta] = \int d^4x \alpha \sqrt{g} \left(R + K_{ab} K^{ab} - K^2 \right). \quad (8)$$

The extrinsic curvature is defined by

$$K_{ab} \equiv -\frac{1}{2\alpha} \left(\dot{g}_{ab} - 2D_{(a}\beta_{b)} \right), \quad (9)$$

and D_a denotes the spatial covariant derivative. The Hamiltonian can be derived in a straightforward fashion if one recognizes that the action does not contain time derivatives of the lapse and shift. Time derivatives of the spatial metric appear through the combination K_{ab} . Thus, we introduce the momentum

$$P^{ab} \equiv \frac{\partial \mathcal{L}}{\partial \dot{g}_{ab}} = \sqrt{g} \left(K^{ab} - K^{ab} \right), \quad (10)$$

where the Lagrangian density \mathcal{L} is the integrand of the action. This definition can be inverted for \dot{g}_{ab} as a function of P^{ab} and used to define the Hamiltonian: $H \equiv \int d^3x (P^{ab} \dot{g}_{ab} - \mathcal{L})$. This yields the ADM Hamiltonian

$$H = \int d^3x (\alpha \mathcal{H} + \beta^a \mathcal{M}_a), \quad (11)$$

where

$$\mathcal{H} \equiv \frac{1}{\sqrt{g}} \left(P^{ab} P_{ab} - P^2/2 \right) - \sqrt{g} R, \quad (12a)$$

$$\mathcal{M}_a \equiv -2D_b P_a^b \quad (12b)$$

are the Hamiltonian and momentum constraints.

In the analysis above the lapse and shift are treated as non dynamical fields. They appear in the Hamiltonian form of the action,

$$S[g, P, \alpha, \beta] = \int_{t_i}^{t_f} dt \int d^3x \left\{ P^{ab} \dot{g}_{ab} - \alpha \mathcal{H} - \beta^a \mathcal{M}_a \right\}, \quad (13)$$

as undetermined multipliers. Here t_i and t_f are the initial and final times. Extremization of S with respect to α and β^a yields the constraints $\mathcal{H} = 0$ and $\mathcal{M}_a = 0$.

We can enlarge the ADM formulation to include the lapse and shift as dynamical variables.[7, 12] Consider again the action (8). In addition to the momentum P^{ab} conjugate to the spatial metric, we also define conjugate variables for the lapse and shift:

$$\pi \equiv \frac{\partial \mathcal{L}}{\partial \dot{\alpha}} = 0, \quad (14a)$$

$$\rho_a \equiv \frac{\partial \mathcal{L}}{\partial \dot{\beta}^a} = 0. \quad (14b)$$

This leads to primary constraints $\pi = 0$ and $\rho_a = 0$. The resulting Hamiltonian is not unique; it is only determined to within the addition of arbitrary multiples of the constraints:

$$H = \int d^3x (\alpha \mathcal{H} + \beta^a \mathcal{M}_a + \Lambda \pi + \Omega^a \rho_a). \quad (15)$$

The coefficients Λ and Ω^a appear as undetermined multipliers in the action, which now reads

$$S[g, P, \alpha, \pi, \beta, \rho, \Lambda, \Omega] = \int_{t_i}^{t_f} dt \int d^3x \left\{ P^{ab} \dot{g}_{ab} + \pi \dot{\alpha} + \rho_a \dot{\beta}^a - \alpha \mathcal{H} - \beta^a \mathcal{M}_a - \Lambda \pi - \Omega^a \rho_a \right\}. \quad (16)$$

The equations of motion, $\delta S = 0$, are²

$$\dot{g}_{ab} = \mathcal{L}_\beta g_{ab} + \frac{\alpha}{\sqrt{g}} (2P_{ab} - P g_{ab}), \quad (17a)$$

$$\begin{aligned} \dot{P}^{ab} = \mathcal{L}_\beta P^{ab} + \frac{\alpha}{\sqrt{g}} (\delta_c^a \delta_d^b - g^{ab} g_{cd}/4) (P P^{cd} - 2P^{ce} P_e^d) \\ - \alpha \sqrt{g} G^{ab} + \sqrt{g} (D^a D^b \alpha - g^{ab} D_c D^c \alpha), \end{aligned} \quad (17b)$$

$$\dot{\alpha} = \Lambda, \quad (17c)$$

$$\dot{\pi} = -\mathcal{H}, \quad (17d)$$

$$\dot{\beta}^a = \Omega^a, \quad (17e)$$

$$\dot{\rho}_a = -\mathcal{M}_a, \quad (17f)$$

$$\pi = 0, \quad (17g)$$

$$\rho_a = 0, \quad (17h)$$

where G^{ab} denotes the spatial Einstein tensor and \mathcal{L}_β is the Lie derivative with respect to β^a .

The equations above must hold for each time $t_i \leq t \leq t_f$. They are equivalent to the Einstein equations supplemented with evolution equations for the lapse function and shift vector. In particular, observe that π and ρ_a must vanish for all time by Eqs. (17g, 17h). It follows that the time derivatives of π and ρ_a must vanish. In turn, Eqs. (17d, 17f) imply that the usual Hamiltonian and momentum constraints are zero. Eqs. (17a, 17b) are the familiar ADM evolution equations, and Eqs. (17c, 17e) supply evolution equations for the lapse and shift.

The evolution equations (17a–f) are Hamilton's equations derived from the Hamiltonian (15). The time derivative of any function F of the canonical variables is $\dot{F} = \{F, H\}$ where the fundamental Poisson brackets relations are defined by $\{g_{ab}(x), P^{cd}(x')\} = \delta_{(a}^{(c} \delta_{b)}^{d)} \delta^3(x, x')$, $\{\alpha(x), \pi(x')\} = \delta^3(x, x')$, and $\{\beta^a(x), \rho_b(x')\} = \delta_b^a \delta^3(x, x')$. We can interpret Hamilton's equations as an initial value problem by following Dirac's reasoning for constrained Hamiltonian systems.[13] The initial data are chosen such that the primary constraints π and ρ_a vanish at the initial time. According to Eqs. (17d, 17f), these constraints will remain zero as long as \mathcal{H} and \mathcal{M}_a are constrained to vanish as well. Thus we impose $\mathcal{H} = 0$ and $\mathcal{M}_a = 0$ as secondary constraints. The complete set of constraints, $\pi = 0$, $\rho_a = 0$, $\mathcal{H} = 0$, and $\mathcal{M}_a = 0$ is first class.

² Throughout this paper I ignore the issues that arise when space has boundaries.[21, 3]

4 Hyperbolicity of ADM with dynamical lapse and shift

Hamilton's equations (17a–f) are equivalent to the ADM equations plus evolution equations $\dot{\alpha} = \Lambda$, $\dot{\beta}^a = \Omega^a$ for the lapse and shift. Let us analyze the level of hyperbolicity of these PDE's. The principal parts of the \dot{q} equations are the terms that are proportional to p 's or first spatial derivatives of q 's. The principal parts of the \dot{p} equations are the terms that are proportional to first spatial derivatives of p 's or second spatial derivatives of q 's. Thus, we find

$$\hat{\partial}_0 g_{ab} \cong 2g_{c(a} \partial_{b)} \beta^c + \frac{\alpha}{\sqrt{g}} (2P_{ab} - P g_{ab}), \quad (18a)$$

$$\begin{aligned} \hat{\partial}_0 P^{ab} \cong & \frac{\alpha \sqrt{g}}{2} g^{ac} g^{bd} g^{ef} (\partial_e \partial_f g_{cd} - 2\partial_e \partial_{(c} g_{d)f} + \partial_c \partial_d g_{ef}) \\ & + \frac{\alpha \sqrt{g}}{2} g^{ab} g^{cd} g^{ef} (\partial_c \partial_e g_{df} - \partial_c \partial_d g_{ef}) \\ & + \sqrt{g} (g^{ac} g^{bd} - g^{ab} g^{cd}) \partial_c \partial_d \alpha, \end{aligned} \quad (18b)$$

$$\hat{\partial}_0 \alpha \cong -\beta^a \partial_a \alpha, \quad (18c)$$

$$\hat{\partial}_0 \pi \cong \sqrt{g} g^{ab} g^{cd} (\partial_a \partial_c g_{bd} - \partial_a \partial_b g_{cd}) - \beta^a \partial_a \pi, \quad (18d)$$

$$\hat{\partial}_0 \beta^a \cong -\beta^b \partial_b \beta^a, \quad (18e)$$

$$\hat{\partial}_0 \rho_a \cong 2g_{ac} \partial_b P^{bc} - \beta^b \partial_b \rho_a, \quad (18f)$$

where the symbol \cong is used to denote equality up to lower order (non principal) terms. These equations have been expressed in terms of the operator $\hat{\partial}_0 \equiv \partial_t - \beta^a \partial_a$ so that the characteristic speeds (the eigenvalues of the principal symbol) are defined with respect to observers who are at rest in the spacelike slices.

We now construct the eigenvalue problem $\mu v = A v$ for the principal symbol A . The principal symbol is found from the coefficients on the right-hand sides of Eqs. (18). These coefficients are divided by a factor of α so that the characteristic speeds will be expressed in terms of proper time rather than coordinate time. The result is

$$\mu \bar{g}_{ab} = \frac{2}{\alpha} n_{(a} \bar{\beta}_{b)} + \frac{1}{\sqrt{g}} [2\bar{P}_{ab} - g_{ab} (\bar{P}_{nn} + \bar{P}_{AB} \delta^{AB})], \quad (19a)$$

$$\begin{aligned} \mu \bar{P}_{ab} = & \frac{\sqrt{g}}{2} [\bar{g}_{ab} - 2n_{(a} \bar{g}_{b)n} + n_a n_b (\bar{g}_{nn} + \bar{g}_{AB} \delta^{AB}) - g_{ab} \bar{g}_{AB} \delta^{AB}] \\ & - \frac{\sqrt{g}}{\alpha} (g_{ab} - n_a n_b) \bar{\alpha}, \end{aligned} \quad (19b)$$

$$\mu \bar{\alpha} = -(\beta \cdot n / \alpha) \bar{\alpha}, \quad (19c)$$

$$\mu \bar{\pi} = -\frac{\sqrt{g}}{\alpha} \bar{g}_{AB} \delta^{AB} - (\beta \cdot n / \alpha) \bar{\pi}, \quad (19d)$$

$$\mu \bar{\beta}_a = -(\beta \cdot n / \alpha) \bar{\beta}_a, \quad (19e)$$

$$\mu \bar{\rho}_a = \frac{2}{\alpha} \bar{P}_{na} - (\beta \cdot n / \alpha) \bar{\rho}_a. \quad (19f)$$

where μ is the eigenvalue and $v = (\bar{g}_{ab}, \bar{P}_{ab}, \bar{\alpha}, \bar{\pi}, \bar{\beta}_a, \bar{\rho}_a)^T$ is the eigenvector. In these equations n^a is normalized with respect to the spatial metric, $n^a g_{ab} n^b = 1$, and a subscript n denotes contraction with n^a . We have also introduced an orthonormal diad e_A^a with $A = 1, 2$ in the subspace orthogonal to n_a . That is, $n_a e_A^a = 0$ and $e_A^a g_{ab} e_B^b = \delta_{AB}$. A subscript A on a tensor (such as the metric g_{ab} or momentum P_{ab}) denotes contraction with e_A^a .

The eigenvalue problem (19) splits into scalar, vector and trace-free tensor blocks with respect to rotations about the normal direction n_a . The scalar block is

$$\mu \bar{g}_{nn} = \frac{2}{\alpha} \bar{\beta}_n + \frac{1}{\sqrt{g}} (\bar{P}_{nn} - \bar{P}_{AB} \delta^{AB}), \quad (20a)$$

$$\mu \bar{g}_{AB} \delta^{AB} = -\frac{2}{\sqrt{g}} \bar{P}_{nn}, \quad (20b)$$

$$\mu \bar{P}_{nn} = 0, \quad (20c)$$

$$\mu \bar{P}_{AB} \delta^{AB} = -\frac{1}{2} \sqrt{g} \bar{g}_{AB} \delta^{AB} - \frac{2}{\alpha} \sqrt{g} \bar{\alpha}, \quad (20d)$$

$$\mu \bar{\alpha} = -(\beta \cdot n / \alpha) \bar{\alpha}, \quad (20e)$$

$$\mu \bar{\pi} = -\frac{\sqrt{g}}{\alpha} \bar{g}_{AB} \delta^{AB} - (\beta \cdot n / \alpha) \bar{\pi}, \quad (20f)$$

$$\mu \bar{\beta}_n = -(\beta \cdot n / \alpha) \bar{\beta}_n, \quad (20g)$$

$$\mu \bar{\rho}_n = \frac{2}{\alpha} \bar{P}_{nn} - (\beta \cdot n / \alpha) \bar{\rho}_n. \quad (20h)$$

This block has eigenvalues 0 and $-(\beta \cdot n / \alpha)$, each with multiplicity 4. There is only one eigenvector with eigenvalue 0, so the eigenvectors are not complete. The vector block is

$$\mu \bar{g}_{nA} = \frac{1}{\alpha} \bar{\beta}_A + \frac{2}{\sqrt{g}} \bar{P}_{nA}, \quad (21a)$$

$$\mu \bar{P}_{nA} = 0, \quad (21b)$$

$$\mu \bar{\beta}_A = -(\beta \cdot n / \alpha) \bar{\beta}_A, \quad (21c)$$

$$\mu \bar{\rho}_A = \frac{2}{\alpha} \bar{P}_{nA} - (\beta \cdot n / \alpha) \bar{\rho}_A. \quad (21d)$$

It has eigenvalues 0 and $-(\beta \cdot n / \alpha)$, each with multiplicity 2. There is only one eigenvector with eigenvalue 0, so the eigenvectors are not complete. Finally, the trace-free tensor block is

$$\mu \bar{g}_{AB}^{tf} = \frac{2}{\sqrt{g}} \bar{P}_{AB}^{tf}, \quad (22a)$$

$$\mu \bar{P}_{AB}^{tf} = \frac{\sqrt{g}}{2} \bar{g}_{AB}^{tf}. \quad (22b)$$

This block has eigenvalues ± 1 and a complete set of eigenvectors. Because the eigenvectors for the scalar and vector blocks are not complete, the system (17) is weakly hyperbolic.

5 Extending the ADM formulation

In the previous section we modified the ADM Hamiltonian formulation of general relativity so that the lapse function α and shift vector β^a are treated as dynamical variables. Their canonical conjugates are denoted π and ρ_a , respectively. The undetermined multipliers for the constraints $\pi = 0$, $\rho_a = 0$ are Λ and Ω^a . These multipliers are freely specifiable functions of space and time. They determine the slicing and spatial coordinate conditions through the equations of motion $\dot{\alpha} = \Lambda$ and $\dot{\beta}^a = \Omega^a$.

Here is the key observation. The histories that extremize the action are unchanged if we replace the multipliers by $\Lambda \rightarrow \Lambda + \hat{\Lambda}$ and $\Omega^a \rightarrow \Omega^a + \hat{\Omega}^a$, where $\hat{\Lambda}$ and $\hat{\Omega}^a$ are quasilinear functions of the canonical variables. By quasilinear, I mean that the principal parts of $\hat{\Lambda}$

and $\hat{\Omega}^a$ are linear in the momenta (P^{ab} , π and ρ_a) and first spatial derivatives of the coordinates ($\partial_c g_{ab}$, $\partial_c \alpha$ and $\partial_c \beta^a$) with coefficients that depend on the coordinates. With these replacements the action becomes

$$S[g, P, \alpha, \pi, \beta, \rho, \Lambda, \Omega] = \int_{t_i}^{t_f} dt \int d^3x \left\{ P^{ab} \dot{g}_{ab} + \pi \dot{\alpha} + \rho_a \dot{\beta}^a - \alpha \mathcal{H} - \beta^a \mathcal{M}_a - (\Lambda + \hat{\Lambda})\pi - (\Omega^a + \hat{\Omega}^a)\rho_a \right\}, \quad (23)$$

and the Hamiltonian is

$$H = \int d^3x (\alpha \mathcal{H} + \beta^a \mathcal{M}_a + (\Lambda + \hat{\Lambda})\pi + (\Omega^a + \hat{\Omega}^a)\rho_a). \quad (24)$$

The solutions to the equations of motion are unaltered because the extra terms are proportional to the constraints $\pi = 0$, $\rho_a = 0$. In the Hamiltonian formulation we can dispense with the original multipliers Λ and Ω^a ; that is, these quantities can be absorbed into the functions $\hat{\Lambda}$ and $\hat{\Omega}^a$.

With $\hat{\Lambda}$ and $\hat{\Omega}^a$ restricted to be quasilinear in the momenta and first spatial derivatives of the coordinates, the equations of motion become

$$\dot{g}_{ab} = \mathcal{L}_\beta g_{ab} + \frac{\alpha}{\sqrt{g}}(2P_{ab} - P g_{ab}) + \frac{\partial \hat{\Lambda}}{\partial P^{ab}} \pi + \frac{\partial \hat{\Omega}^c}{\partial P^{ab}} \rho_c, \quad (25a)$$

$$\begin{aligned} \dot{P}^{ab} = & \mathcal{L}_\beta P^{ab} + \frac{\alpha}{\sqrt{g}}(\delta_c^a \delta_d^b - g^{ab} g_{cd}/4)(P P^{cd} - 2P^{ce} P_e^d) \\ & - \alpha \sqrt{g} G^{ab} + \sqrt{g}(D^a D^b \alpha - g^{ab} D_c D^c \alpha) \\ & - \frac{\partial \hat{\Lambda}}{\partial g_{ab}} \pi - \frac{\partial \hat{\Omega}^c}{\partial g_{ab}} \rho_c + \partial_d \left(\frac{\partial \hat{\Lambda}}{\partial (\partial_d g_{ab})} \pi \right) + \partial_d \left(\frac{\partial \hat{\Omega}^c}{\partial (\partial_d g_{ab})} \rho_c \right), \end{aligned} \quad (25b)$$

$$\dot{\alpha} = \Lambda + \hat{\Lambda} + \frac{\partial \hat{\Lambda}}{\partial \pi} \pi + \frac{\partial \hat{\Omega}^c}{\partial \pi} \rho_c, \quad (25c)$$

$$\dot{\pi} = -\mathcal{H} - \frac{\partial \hat{\Lambda}}{\partial \alpha} \pi - \frac{\partial \hat{\Omega}^c}{\partial \alpha} \rho_c + \partial_d \left(\frac{\partial \hat{\Lambda}}{\partial (\partial_d \alpha)} \pi \right) + \partial_d \left(\frac{\partial \hat{\Omega}^c}{\partial (\partial_d \alpha)} \rho_c \right), \quad (25d)$$

$$\dot{\beta}^a = \Omega^a + \hat{\Omega}^a + \frac{\partial \hat{\Lambda}}{\partial \rho_a} \pi + \frac{\partial \hat{\Omega}^c}{\partial \rho_a} \rho_c, \quad (25e)$$

$$\dot{\rho}_a = -\mathcal{M}_a - \frac{\partial \hat{\Lambda}}{\partial \beta^a} \pi - \frac{\partial \hat{\Omega}^c}{\partial \beta^a} \rho_c + \partial_d \left(\frac{\partial \hat{\Lambda}}{\partial (\partial_d \beta^a)} \pi \right) + \partial_d \left(\frac{\partial \hat{\Omega}^c}{\partial (\partial_d \beta^a)} \rho_c \right), \quad (25f)$$

$$\pi = 0, \quad (25g)$$

$$\rho_a = 0, \quad (25h)$$

Equations (25a, 25b) are the usual ADM equations apart from terms proportional to the constraints, $\pi = 0$ and $\rho_a = 0$. The equations that govern the slicing and spatial coordinates are generalized by the presence of the functions $\hat{\Lambda}$ and $\hat{\Omega}^a$. Apart from terms that vanish with the constraints $\pi = 0$, $\rho_a = 0$, the evolution equation for the lapse becomes $\dot{\alpha} = \Lambda + \hat{\Lambda}$ and the evolution equation for the shift becomes $\dot{\beta}^a = \Omega^a + \hat{\Omega}^a$. The equations for $\dot{\pi}$ and $\dot{\rho}_a$ are modified, but once again we see that the complete set of constraints $\pi = 0$, $\rho_a = 0$, $\mathcal{H} = 0$, and $\mathcal{M}_a = 0$ is first class.

In principle we can choose $\hat{\Lambda}$ and $\hat{\Omega}^a$ to be any set of quasilinear functions of the canonical variables. In practice we might want $\hat{\Lambda}$ and $\hat{\Omega}^a$ to satisfy certain transformation properties. For example we can restrict $\hat{\Lambda}$ to be a scalar and $\hat{\Omega}^a$ to be a contravariant vector under spatial diffeomorphisms. This allows us to maintain a geometrical interpretation of the equations of motion. In particular this allows us to prepare and evolve identical geometrical data using different spatial coordinate systems.

Another property that can be imposed on the formalism is reparametrization invariance.[13] This is invariance under a change of coordinate labels t for the constant time slices. Consider the infinitesimal transformation $t \rightarrow t - \varepsilon(t)$. In the usual ADM system, the variables g_{ab} and P^{ab} transform as scalars under time reparametrization: $\delta g_{ab} = \varepsilon \dot{g}_{ab}$ and $\delta P^{ab} = \varepsilon \dot{P}^{ab}$. The time derivative of the metric, \dot{g}_{ab} , transforms as a covariant vector. In one dimension a covariant vector transforms in the same way as a scalar density of weight +1. It follows that the term $P^{ab} \dot{g}_{ab}$ that appears in the action is a weight +1 scalar density. For reparametrization invariance to hold, the integrand of the action should transform as a weight +1 scalar density since it is integrated over t . In particular the lapse function α and shift vector β^a , which multiply the scalars \mathcal{H} and \mathcal{M}_a , must transform as scalar densities of weight +1.

Observe that the time derivatives $\dot{\alpha}$ and $\dot{\beta}^a$ are constructed from coordinate derivatives of scalar densities and, as a consequence, these quantities do not transform as tensors or tensor densities. We need to replace the coordinate derivatives (dots) with covariant derivatives. We can do this by choosing a background metric for the time direction. This should be viewed as part of the gauge fixing process. Now, the physical metric for the time manifold is α^2 , so let $\tilde{\alpha}^2$ denote the background metric. The covariant derivative built from $\tilde{\alpha}^2$, acting on the densities α and β^a , is defined by

$$\tilde{\alpha} \equiv \dot{\alpha} - (\dot{\tilde{\alpha}}/\tilde{\alpha})\alpha, \quad (26a)$$

$$\tilde{\beta}^a \equiv \dot{\beta}^a - (\dot{\tilde{\alpha}}/\tilde{\alpha})\beta^a. \quad (26b)$$

The extra terms needed for reparametrization invariance can be built into the action or Hamiltonian by including a term $(\tilde{\alpha}/\tilde{\alpha})\alpha$ in the function $\hat{\Lambda}$ and a term $(\tilde{\alpha}/\tilde{\alpha})\beta^a$ in the function $\hat{\Omega}^a$.

With the appropriate terms included in $\hat{\Lambda}$ and $\hat{\Omega}^a$, the time derivatives of the lapse and shift appear in the action only in the combinations $\pi \tilde{\alpha}$ and $\rho_a \tilde{\beta}^a$. Since $\tilde{\alpha}$ and $\tilde{\beta}^a$ are covariant vector densities of weight +1, we see that π and ρ_a must transform as contravariant vectors with no density weight. In one dimension, contravariant vectors transform in the same way as a scalar density of weight -1. We will consider π and ρ_a to be scalar densities of weight -1 under time reparametrization. It follows that, apart from the terms $(\tilde{\alpha}/\tilde{\alpha})\alpha$ and $(\tilde{\alpha}/\tilde{\alpha})\beta^a$, the multipliers $\Lambda + \hat{\Lambda}$ and $\Omega^a + \hat{\Omega}^a$ should transform as scalar densities of weight +2.

We have now established the rules for adding terms to the functions $\hat{\Lambda}$ and $\hat{\Omega}^a$ such that the resulting formulation is invariant under time reparametrizations: these terms must be weight +2 densities built from the scalars g_{ab} , P^{ab} , the weight +1 densities α , β^a , and the weight -1 densities π , ρ_a . We can also insist that $\hat{\Lambda}$ and $\hat{\Omega}^a$ should be, respectively, a scalar and a contravariant vector under spatial diffeomorphisms. With these properties in mind, a fairly general form for $\hat{\Lambda}$ is

$$\hat{\Lambda} = (\tilde{\alpha}/\tilde{\alpha})\alpha + \beta^a D_a \alpha - C_1 \frac{\alpha^2}{\sqrt{g}} P + C_4 \frac{\alpha^3}{\sqrt{g}} \pi. \quad (27)$$

The first term is required for reparametrization invariance. The second term will allow us to combine shift vector terms into a Lie derivative \mathcal{L}_β acting on α . The terms multiplied by constants C_1 and C_4 are principal terms that will affect the hyperbolicity of the resulting

equations. There are other principal terms that one can add, such as $\alpha^2 \beta^a \rho_a / \sqrt{g}$, but the form above will be general enough for present purposes. There are also lower order terms that one can add to $\hat{\Lambda}$.

For $\hat{\Omega}^a$ we must construct a spatial vector that [apart from the term $(\dot{\alpha}/\tilde{\alpha})\beta^a$] transforms as a weight +2 density under time reparametrizations. There are several obvious ways to construct a spatial vector from the canonical variables at hand. There are some possibilities that are not so obvious. Recall that the difference of two connections is a tensor. Thus, the combination $\Gamma_{bc}^a - \tilde{\Gamma}_{bc}^a$ is a spatial tensor if Γ_{bc}^a are the Christoffel symbols built from the physical metric g_{ab} and $\tilde{\Gamma}_{bc}^a$ are the Christoffel symbols built from a background metric \tilde{g}_{ab} . In setting up a numerical calculation, for example, on a logically Cartesian grid, we can take \tilde{g}_{ab} to be the flat metric in Cartesian coordinates. Again, we view the introduction of the background structure \tilde{g}_{ab} as part of the gauge fixing process.

The general form for $\hat{\Omega}^a$ that we will consider is

$$\begin{aligned} \hat{\Omega}^a = & (\dot{\alpha}/\tilde{\alpha})\beta^a + \beta^b \tilde{D}_b \beta^a + C_2 \alpha^2 (\Gamma_{bc}^a - \tilde{\Gamma}_{bc}^a) g^{bc} \\ & + C_3 \alpha^2 (\Gamma_{cb}^c - \tilde{\Gamma}_{cb}^c) g^{ab} - C_5 \alpha D^a \alpha - C_6 \frac{\alpha^3}{\sqrt{g}} \rho^a . \end{aligned} \quad (28)$$

where \tilde{D}_a is the covariant derivative compatible with \tilde{g}_{ab} . The first term is required for time reparametrization invariance. The second term will allow us to combine time derivatives and shift vector terms into the operator $\hat{\partial}_0$ in the principle parts of the equations for β^a and ρ_a . The remaining terms will modify the principal parts of the equations of motion and can affect the hyperbolicity of the system. There are other principal terms that we could add to $\hat{\Omega}^a$, such as $\alpha^2 \pi \beta^a / \sqrt{g}$ or $\alpha P^{ab} \beta_b / \sqrt{g}$. We can also add lower order terms.

With these expressions for $\hat{\Lambda}$ and $\hat{\Omega}^a$, we find the following equations of motion by varying the action (23):

$$\dot{g}_{ab} = \mathcal{L}_\beta g_{ab} + \frac{\alpha}{\sqrt{g}} (2P_{ab} - P g_{ab}) - C_1 \frac{\alpha^2}{\sqrt{g}} \pi g_{ab} , \quad (29a)$$

$$\begin{aligned} \dot{P}^{ab} = & \mathcal{L}_\beta P^{ab} + \frac{\alpha}{\sqrt{g}} (\delta_c^a \delta_d^b - g^{ab} g_{cd}/4) (P P^{cd} - 2P^{ce} P_e^d) - \alpha \sqrt{g} G^{ab} + \\ & \sqrt{g} (D^a D^b \alpha - g^{ab} D_c D^c \alpha) + C_1 \frac{\alpha^2}{\sqrt{g}} (P^{ab} - P g^{ab}/2) \pi + C_4 \frac{\alpha^3}{2\sqrt{g}} \pi^2 g^{ab} \\ & + C_2 D^{(a} (\rho^{b)} \alpha^2) - C_5 \alpha P^{(a} D^{b)} \alpha + \frac{1}{2} (C_3 - C_2) D_c (\alpha^2 P^c) g^{ab} \\ & + C_2 \alpha^2 P_e (\Gamma_{cd}^e - \tilde{\Gamma}_{cd}^e) g^{ac} g^{bd} + C_3 \alpha^2 (\Gamma_{cd}^d - \tilde{\Gamma}_{cd}^d) P^{(a} g^{b)c} \\ & - C_6 \frac{\alpha^3}{\sqrt{g}} (\rho^a \rho^b + \rho_c \rho^c g^{ab}/2) , \end{aligned} \quad (29b)$$

$$\dot{\alpha} = \mathcal{L}_\beta \alpha + \Lambda - C_1 \frac{\alpha^2}{\sqrt{g}} P + 2C_4 \frac{\alpha^3}{\sqrt{g}} \pi , \quad (29c)$$

$$\begin{aligned} \dot{\pi} = & \mathcal{L}_\beta \pi - \mathcal{H} + 2C_1 \frac{\alpha}{\sqrt{g}} P \pi - 3C_4 \frac{\alpha^2}{\sqrt{g}} \pi^2 - 2C_2 \alpha (\Gamma_{bc}^a - \tilde{\Gamma}_{bc}^a) g^{bc} \rho_a \\ & - 2C_3 \alpha (\Gamma_{ab}^b - \tilde{\Gamma}_{ab}^b) \rho^a - C_5 \alpha D_a \rho^a + 3C_6 \frac{\alpha^2}{\sqrt{g}} \rho_a \rho^a , \end{aligned} \quad (29d)$$

$$\dot{\beta}^a = \beta^b \tilde{D}_b \beta^a + \Omega^a + C_2 \alpha^2 (\Gamma_{bc}^a - \tilde{\Gamma}_{bc}^a) g^{bc} + C_3 \alpha^2 (\Gamma_{bc}^c - \tilde{\Gamma}_{bc}^c) g^{ab}$$

$$-C_5 \alpha D^a \alpha - 2C_6 \frac{\alpha^3}{\sqrt{g}} \rho^a, \quad (29e)$$

$$\dot{\rho}_a = \tilde{D}_b (\beta^b \rho_a) - \rho_b \tilde{D}_a \beta^b - \mathcal{M}_a - \pi D_a \alpha, \quad (29f)$$

$$\pi = 0, \quad (29g)$$

$$\rho_a = 0. \quad (29h)$$

These equations are generally covariant under spatial diffeomorphisms and time reparametrizations. Equations (29a–f) are generated through the Poisson brackets by the Hamiltonian (24). Note that for g_{ab} and P^{ab} , which are scalars under time reparametrization, the covariant time derivative (circle) is equivalent to a coordinate time derivative (dot).

6 Hyperbolicity of the Extended ADM formulation

The principal parts of the extended ADM evolution equations (29a–f) are:

$$\hat{\partial}_0 g_{ab} \cong 2g_{c(a} \partial_{b)} \beta^c + \frac{\alpha}{\sqrt{g}} (2P_{ab} - P g_{ab}) - C_1 \frac{\alpha^2}{\sqrt{g}} \pi g_{ab}, \quad (30a)$$

$$\begin{aligned} \hat{\partial}_0 P^{ab} \cong & \frac{\alpha \sqrt{g}}{2} g^{ac} g^{bd} g^{ef} (\partial_e \partial_f g_{cd} - 2\partial_e \partial_{(c} g_{d)f} + \partial_c \partial_d g_{ef}) \\ & + \frac{\alpha \sqrt{g}}{2} g^{ab} g^{cd} g^{ef} (\partial_c \partial_e g_{df} - \partial_c \partial_d g_{ef}) + \sqrt{g} (g^{ac} g^{bd} - g^{ab} g^{cd}) \partial_c \partial_d \alpha \\ & + \alpha^2 \left[C_2 g^{c(a} g^{b)d} + (C_3 - C_2) g^{ab} g^{cd} / 2 \right] \partial_c \rho_d, \end{aligned} \quad (30b)$$

$$\hat{\partial}_0 \alpha \cong -C_1 \frac{\alpha^2}{\sqrt{g}} P + 2C_4 \frac{\alpha^3}{\sqrt{g}} \pi, \quad (30c)$$

$$\hat{\partial}_0 \pi \cong \sqrt{g} g^{ab} g^{cd} (\partial_a \partial_c g_{bd} - \partial_a \partial_b g_{cd}) - C_5 \alpha g^{ab} \partial_a \rho_b, \quad (30d)$$

$$\begin{aligned} \hat{\partial}_0 \beta^a \cong & \alpha^2 \left[C_2 g^{ac} g^{bd} + (C_3 - C_2) g^{ab} g^{cd} / 2 \right] \partial_b g_{cd} \\ & - C_5 \alpha g^{ab} \partial_b \alpha - 2C_6 \frac{\alpha^3}{\sqrt{g}} \rho^a, \end{aligned} \quad (30e)$$

$$\hat{\partial}_0 \rho_a \cong 2g_{ac} \partial_b P^{bc}, \quad (30f)$$

From here it is easy to construct the eigenvalue problem for the principal symbol. The symbol decomposes into scalar, vector, and trace-free tensor blocks under rotations orthogonal to the normal vector $n_a \equiv k_a / |k|$. For the scalar sector, we find

$$\mu \bar{g}_{nm} = \frac{2}{\alpha} \bar{\beta}_n + \frac{1}{\sqrt{g}} (\bar{P}_{nm} - \bar{P}_{AB} \delta^{AB}) - C_1 \frac{\alpha}{\sqrt{g}} \bar{\pi}, \quad (31a)$$

$$\mu \bar{g}_{AB} \delta^{AB} = -\frac{2}{\sqrt{g}} \bar{P}_{nm} - 2C_1 \frac{\alpha}{\sqrt{g}} \bar{\pi}, \quad (31b)$$

$$\mu \bar{P}_{nm} = \frac{\alpha}{2} (C_3 + C_2) \bar{\rho}_n, \quad (31c)$$

$$\mu \bar{P}_{AB} \delta^{AB} = -\frac{1}{2} \sqrt{g} \bar{g}_{AB} \delta^{AB} - \frac{2}{\alpha} \sqrt{g} \bar{\alpha} + \alpha (C_3 - C_2) \bar{\rho}_n, \quad (31d)$$

$$\mu \bar{\alpha} = -C_1 \frac{\alpha}{\sqrt{g}} (\bar{P}_{nm} + \bar{P}_{AB} \delta^{AB}) + 2C_4 \frac{\alpha^2}{\sqrt{g}} \bar{\pi}, \quad (31e)$$

$$\mu \bar{\pi} = -\frac{\sqrt{g}}{\alpha} \bar{g}_{AB} \delta^{AB} - C_5 \bar{\rho}_n, \quad (31f)$$

$$\mu \bar{\beta}_n = \frac{\alpha}{2} (C_3 + C_2) \bar{g}_{nn} + \frac{\alpha}{2} (C_3 - C_2) \bar{g}_{AB} \delta^{AB} - C_5 \bar{\alpha} - 2C_6 \frac{\alpha^2}{\sqrt{g}} \bar{\rho}_n, \quad (31g)$$

$$\mu \bar{\rho}_n = \frac{2}{\alpha} \bar{P}_{nn}. \quad (31h)$$

Again, the subscripts n and A denote contraction with n^a and e_A^a , respectively. The vector block is

$$\mu \bar{g}_{nA} = \frac{1}{\alpha} \bar{\beta}_A + \frac{2}{\sqrt{g}} \bar{P}_{nA}, \quad (32a)$$

$$\mu \bar{P}_{nA} = \frac{\alpha}{2} C_2 \bar{\rho}_A, \quad (32b)$$

$$\mu \bar{\beta}_A = C_2 \alpha \bar{g}_{nA} - 2C_6 \frac{\alpha^2}{\sqrt{g}} \bar{\rho}_A, \quad (32c)$$

$$\mu \bar{\rho}_A = \frac{2}{\alpha} \bar{P}_{nA}, \quad (32d)$$

and the trace-free tensor block is unmodified from before:

$$\mu \bar{g}_{AB}^{tf} = \frac{2}{\sqrt{g}} \bar{P}_{AB}^{tf}, \quad (33a)$$

$$\mu \bar{P}_{AB}^{tf} = \frac{\sqrt{g}}{2} \bar{g}_{AB}^{tf}. \quad (33b)$$

The eigenvalues for the scalar block are $\pm\sqrt{2C_1}$ and $\pm\sqrt{C_2+C_3}$, each with multiplicity two. The eigenvalues for the vector block are $\pm\sqrt{C_2}$, each with multiplicity two. For the tensor block the eigenvalues are ± 1 and the eigenvectors are complete.

The eigenvalues are the characteristic speeds with respect to observers at rest in the space-like hypersurfaces. Let us choose the C 's such that the characteristics are ± 1 , that is, along the physical light cone. Then we must have $C_1 = 1/2$, $C_2 = 1$, and $C_3 = 0$. With this choice a careful analysis of the scalar block shows that the eigenvectors are complete only if $C_4 = 1/8$, $C_5 = 1$, and $C_6 = 1/2$. For these values of the constants the eigenvectors for the vector block are complete as well. It can be shown that with these values for the C 's the principal parts of the equations of motion are equivalent to the generalized harmonic formulation of relativity.[9, 20, 16] This will be discussed elsewhere.[5]

The Hamiltonian system (29) is strongly hyperbolic with the choice of C 's above. For this formulation the gauge conditions are

$$\dot{\bar{\alpha}} = \beta^a D_a \alpha + \Lambda - \frac{\alpha^2}{2} \frac{P}{\sqrt{g}} + \frac{\alpha^3}{4} \frac{\pi}{\sqrt{g}}, \quad (34a)$$

$$\dot{\bar{\beta}}^a = \beta^b \bar{D}_b \beta^a + \Omega^a + \alpha^2 (\Gamma_{bc}^a - \tilde{\Gamma}_{bc}^a) g^{bc} - \alpha D^a \alpha - \frac{\alpha^3}{\sqrt{g}} \rho^a \quad (34b)$$

If we choose Λ and Ω^a to vanish, these gauge equations become

$$\dot{\bar{\alpha}} - \beta^a \partial_a \alpha = \alpha (\dot{\bar{\alpha}}/\bar{\alpha}) - \alpha^2 K, \quad (35a)$$

$$\dot{\bar{\beta}}^a - \beta^b \partial_b \beta^a = \beta^a (\dot{\bar{\beta}}/\bar{\beta}) + \alpha^2 (\Gamma_{bc}^a - \tilde{\Gamma}_{bc}^a) g^{bc} - \alpha g^{ab} \partial_b \alpha, \quad (35b)$$

to within terms that vanish with the constraints $\pi = 0, \rho_a = 0$. Here, $K \equiv P/(2\sqrt{g})$ is the trace of the extrinsic curvature. These gauge conditions agree, to within lower order (non-principal) terms, with the gauge conditions for the generalized harmonic formulation of gravity.[16]

It is not difficult to find choices for the constants (the C 's) that make the system (29) strongly hyperbolic. The trick is to find a strongly hyperbolic system with desirable coordinate and slicing conditions. Conditions similar to (35), supplemented with “gauge driver” equations, have been applied to the binary black hole problem with mixed results.[20, 16, 17] With the BSSN evolution system, the gauge conditions that work well for the binary black hole problem are “1+log slicing”,

$$\dot{\alpha} - \beta^a \partial_a \alpha = -2\alpha K, \quad (36)$$

and “gamma-driver shift”. The gamma-driver condition is usually written as a system of two first order (in time) equations for β^a and an auxiliary field $B^a = 4\beta^a/3$. These equations, along with suitable initial conditions, can be integrated to yield a single first order equation for the shift.[28] Expressed as either two equations or one, the gamma-driver condition depends on the trace (in its lower indices) of the Christoffel symbols built from the conformal metric. In terms of the physical metric, we can write the single-equation form of the gamma-driver shift condition as

$$\dot{\beta}^a - \beta^b \partial_b \beta^a = \frac{3}{4} \sqrt{g}^{2/3} \left[\Gamma_{bc}^a g^{bc} + \frac{1}{3} g^{ab} \Gamma_{bc}^c \right] - \eta \beta^a, \quad (37)$$

where η is a constant parameter. The first term on the right-hand side (apart from the factor of $3/4$) is the trace of the conformal Christoffel symbols.

Let us see if we can find a set of functions $\hat{\Lambda}$ and $\hat{\Omega}^a$ that yield the gauge conditions above, and then ask whether the resulting system is strongly hyperbolic. In this example we dispense with any attempt to construct a formulation that is covariant under spatial diffeomorphisms or time reparametrizations. Comparing Eqs. (25c,25e) with the 1+log slicing condition (36) and the gamma-driver shift condition (37), we find

$$\hat{\Lambda} = \beta^a \partial_a \alpha - \alpha P / \sqrt{g} + F_1 \pi + F_2^a \rho_a, \quad (38a)$$

$$\hat{\Omega}^a = \beta^b \partial_b \beta^a + \frac{3}{4} \sqrt{g}^{2/3} \left[\Gamma_{bc}^a g^{bc} + \frac{1}{3} g^{ab} \Gamma_{bc}^c \right] - \eta \beta^a + F_3^a \pi + F_4^{ab} \rho_b, \quad (38b)$$

where the F 's are functions of the coordinates g_{ab} , α , and β^a (and not their spatial derivatives). In this case the analysis of hyperbolicity is complicated by the fact that the scalar and vector blocks of the principal symbol are coupled by the terms F_2^a , F_3^a , and F_4^{ab} . The combined scalar/vector block has real eigenvalues, but completeness of the eigenvectors can be achieved only if F_4^{ab} depends on the normal direction $n_a \equiv k_a/|k|$. This is not acceptable; the Hamiltonian cannot depend on the propagation direction of a perturbative solution. We conclude that there is no Hamiltonian of the form (24) that yields strongly hyperbolic equations with 1+log slicing (36) and gamma-driver shift (37).

7 Concluding Remarks

This paper outlines a procedure for constructing Hamiltonian formulations of Einstein's theory with dynamical gauge conditions and varying levels of hyperbolicity. One can use this procedure as a tool to help identify well posed formulations of the evolution equations that also maintain Hamiltonian and variational structures. The issues of dynamical gauge conditions and hyperbolicity cannot be separated. They are both dictated by the dependence of the

multipliers $\hat{\Lambda}$ and $\hat{\Omega}^a$ on the canonical variables. There are many possibilities that one can explore for this dependence.

The Hamiltonian and variational formulations of general relativity have shaped our perspective and provided deep insights into the theory. In addition, there are a number of practical uses for a Hamiltonian/variational formulation. With an action principle we can pass between spacetime and space-plus-time formulations by adding or removing momentum variables. We can develop a fully first order multisymplectic version of the theory. We can also develop new computational techniques such as variational and symplectic integrators.[18, 4]

One important issue that has not been addressed here is the constraint evolution system. In numerical simulations it is important to control the growth of constraint violations. This might be accomplished in the present framework by including appropriate terms in $\hat{\Lambda}$ and $\hat{\Omega}^a$ to ensure that the constraints are damped. For example, a damping term $-C\pi$ (where C is a positive constant) can be added to the $\dot{\pi}$ equation by including a lower order term $C\alpha$ in $\hat{\Lambda}$. This issue will be explored in more detail elsewhere.[5]

The formalism outlined here can be further extend by introducing dynamical equations for Λ and Ω^a . This is accomplished by introducing momentum variables conjugate to these multipliers. The new momentum variables are primary constraints and are accompanied by a new set of undetermined multipliers. Dynamical equations for Λ and Ω^a are introduced by allowing the new multipliers to depend on the canonical variables. In this way we can construct gauge driver conditions similar to the ones used with the generalized harmonic formulation.[20, 17] We can also allow for gauge conditions that are expressed as systems of PDE's, such as the two-equation version of the gamma-driver shift condition.

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